Surface Pressure

In Ref. 1 an estimate is made of the surface pressure distribution along the plate. Mention will be made here only of the rather surprising similarity of the results obtained to those of Oguchi[6] based on a nonlinear approximation and to the fact that the governing parameter is $\xi = Re_x/M^2$, in agreement with Oguchi's results and the general argument made recently by Talbot.[7]

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Integral Solution for Compressible Laminar Mixing

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An exact solution for laminar mixing of a compressible fluid with Prandtl number one and a viscosity temperature relationship of the form $u \sim T^\omega$ was obtained by Chapman in Ref. 1. This note considers the solution of this problem using the approximate integral technique. The integral solution is of considerable interest since its simplicity commonly affords additional flexibility, although possibly at the expense of accuracy and uncertainty.

The differential equation of motion for similar solutions is[1]

$$-\frac{\zeta}{2}\frac{du^*}{d\zeta} = \frac{d[(u^*T^{*\omega-1})(du^*/d\zeta)]}{d\zeta} \tag{1}$$

where

$$\zeta \equiv \psi/(u_e C \nu_e x)^{1/2}$$

T^* and u^* represent the temperature and velocity nondimensionalized by the freestream values (represented by the subscript e), and ψ is the stream function. The energy equation for Prandtl number one and adiabatic conditions (although adiabatic flow has been assumed the inclusion of heat transfer would not introduce any additional complication) is

$$T^* = 1 + \frac{\gamma-1}{2}M_e^2 - \left(\frac{\gamma-1}{2}\right)M_e^2u^{*2} \tag{2}$$

The boundary conditions for a free layer with a dear-air environment are

$$\begin{aligned} &\text{at } y = \infty: \ \zeta = \infty, \quad u^* = 1 \\ &y = -\infty: \ u^* = 0 \end{aligned} \tag{3}$$

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Chapman and others (e.g., Lykoudis in Ref. 2) have associated $y = -\infty$ with $\zeta = -\infty$. It is not immediately clear that $\zeta \to -\infty$, since $u^* \to 0$ as $y \to -\infty$, resulting in an indeterminant expression. In fact, as will be illustrated, the velocity distribution is such that ζ actually approaches a finite limit. This conclusion is also indicated by the velocity distributions obtained by Chapman.

In order to establish profile approximations required for an integral solution, it is desirable to first consider the series solution of Eq. (1) for large negative values of y. Assume ζ approaches a limit $-\zeta_n$, and express ζ by

$$\zeta = -\zeta_n + \Delta\zeta = -\zeta_n(1-\epsilon) \tag{4}$$

where ϵ is small for ζ near ζ_n. Introducing Eq. (4) into Eq. (1) gives

$$\frac{du^*}{d\epsilon} = \frac{1}{\zeta_n^2}\frac{d[(u^*T^{*\omega-1})(du^*/d\epsilon)]}{d\epsilon} + \epsilon\frac{du^*}{d\epsilon} \tag{5}$$

First, for simplicity, assume $\omega = 1.0$. Equation (5) can first be solved by neglecting the last term for $\epsilon \ll 1.0$ and using the boundary condition, $u^* = 0$ at $\epsilon = 0$. This first-order result then can be employed to approximate the last term for a higher-order solution. This approach can be continued to obtain the solution to any desired order. The result obtained after four iterations is

$$\mu_0^* = \frac{\zeta_n^2}{2}(\epsilon - 0.25\epsilon^2 + 0.01389\epsilon^3 + 0.0017\epsilon^4 + 0.0001\epsilon^5) \tag{6}$$

The subscript 0 is used to refer to this solution of the outer region of the viscous layer. The solution converges very rapidly over the entire range from $0 \le \epsilon \le 1.0$. [This solution could more conveniently be obtained directly from Eq. (1) by assuming a power series solution. However, the formal solution clearly indicates the appropriate type of series required.] Clearly, Eq. (6) shows that, if ζ_n is allowed to become large, u^* will correspondingly become large (exceeding an upper limit of 1.0 demanded by the boundary condition as $y \to \infty$) even for small values of ϵ, thus precluding the possibility that $\zeta_n \to \infty$. In principle, the exact value of ζ_n must be determined by invoking the boundary condition that $u^* \to 1.0$ as $\epsilon \to \infty$. However, the series of Eq. (6) does not converge sufficiently rapidly for large values of ϵ to permit a direct evaluation of ζ_n. Chapman's results of Ref. 3 indicate that $\zeta_n = 1.233$. The values of u^* given by Eq. (6) with $\zeta_n = 1.233$ show excellent agreement with those of Ref. 3 over the range from $0 \le u^* \le 0.90$ (i.e., $-\zeta_n \le \zeta \le 1.5$).

A series solution of Eq. (1) for $\omega \neq 1.0$ can be generated in exactly the same manner by introducing T^* as given by Eq. (2). It was convenient, however, to simply introduce a power series solution similar to Eq. (6) and then evaluate the undetermined coefficients by Eq. (1). Integrating Eq. (1) twice gives [after incorporating Eq. (2)]

$$\frac{1}{2}\int_0^\epsilon u^*d\epsilon = -\frac{A}{2B\omega}[1-(1-Bu^{*2})^\omega] + \int_0^\epsilon [\textstyle\int \epsilon du^*]\, d\epsilon \tag{7}$$

where

$$A \equiv \left(1 + \frac{\gamma-1}{2}M_e^2\right)^{\omega-1}$$

$$B \equiv \frac{[(\gamma-1)/2]\,M_e^2}{1 + [(\gamma-1)/2]\,M_e^2}$$

The second term of Eq. (7) can be expanded in a Taylor's series and then the assumed power series solution introduced to evaluate the undetermined coefficients. The resulting

Table 1 Velocity profile parameters

$\Gamma \times 10^2$	$(1/A)(\zeta_n^2/2)$	u_D^*	λ
0	0.757	0.580	0.330
−2	0.764	0.581	0.328
−3	0.771	0.582	0.325
−4	0.777	0.583	0.322
−6	0.784	0.585	0.320
−8	0.796	0.588	0.315

velocity distribution in the outer region of the shear layer to seventh order in ϵ is

$$u_0^* = (1/A)(\zeta_n^2/2)[\epsilon - 0.25\epsilon^2 + (0.01389 + 1.3333\Gamma)\epsilon^3 + (0.0017 - 1.0833\Gamma)\epsilon^4 + (0.0001 + 0.3428\Gamma + 5.3333\Gamma^2)\epsilon^5 - (0.0462\Gamma + 7.2222\Gamma^2)\epsilon^6 + (0.0034\Gamma + 3.5507\Gamma^2 + 28.4445\Gamma^3)\epsilon^7] \quad (8)$$

where

$$\Gamma \equiv [(\omega - 1)/4]B[(1/A)(\zeta_n^2/2)]^2$$

Obviously, the power series representation of u_0^* becomes more difficult for $\omega \neq 1.0$ (since $T^* \sim u^{*2}$). However, for ω near unity, Γ is small. For $\omega = 0.76$ and $M_e = 5.0$, the velocity distribution given by Eq. (8) for $0 \leq \epsilon \leq 1$ again agrees extremely well with the results of Chapman[1] if the correct ζ_n is introduced. Any degree of accuracy could be obtained by continuing to higher-orders of approximation.

Equation (8) was employed to represent the velocity profile in the outer region of the viscous layer. In order to complete the solution for the free-layer development by the integral technique, it is necessary to match this outer solution with an approximation of the velocity profile in the inner region ($0 \leq \zeta \leq \infty$). The inner region was represented by the simple series

$$u_1^* = 1 - (1 - u_D^*)\left(1 - \frac{\zeta}{\zeta_\delta}\right)^3 = 1 - 1(1 - u_D^*)\left(1 - \lambda\frac{\zeta}{\zeta_n}\right)^3 \quad (9)$$

which satisfies the boundary conditions

$$u^* = 1 \text{ and } \frac{du^*}{d\zeta} = \frac{d^2u^*}{d\zeta^2} = 0 \text{ at } \zeta = \zeta_\delta$$

ζ_δ is taken as the effective inner boundary of the free layer, u_D^* is the value of u^* at $\zeta = 0$, and $\lambda \equiv \zeta_n/\zeta_\delta$. u_D^* and λ of Eq. (9) were related to the constants of Eq. (8) $[(1/A)(\zeta_n/2)$ and $\Gamma]$ by matching the velocity and its first derivative at $\zeta = 0$. These velocity profiles were then inserted into integrated momentum equation to evaluate the unknown ζ_n for any Γ and A. The integrated momentum equation is (expressing the momentum at any x in terms of the initial momentum)

$$\int_0^{\zeta_\delta} u_i^*\, d\zeta + \int_{\zeta_n}^0 u_0^*\, d\zeta = \zeta_\delta \quad (10)$$

The solution of Eq. (10) after introducing u_i^* and u_0^* gives

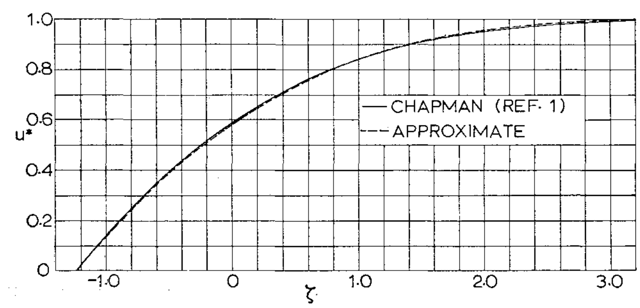

Fig. 1 Comparison of velocity profiles (ω = 1.0).

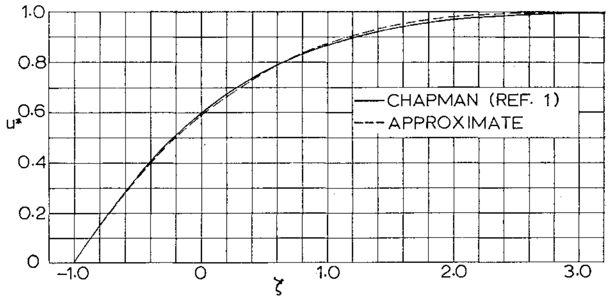

Fig. 2 Comparison of velocity profiles (ω = 0.76 and M_e = 5.0).

the results tabulated in Table 1. These results establish the velocity profiles as well as ζ_n, u_D^*, and λ for specified values of M_e and ω (near unity). A comparison of the velocity distribution expressed by these results with those of Ref. 1 is shown in Fig. 1 for $\omega = 1.0$ (any M_e) and in Fig. 2 for $\omega = 0.76$ and $M_e = 5.0$. The agreement is good in both cases. The velocity u_D^* at $\zeta = 0$ is 0.580 for $\omega = 1.0$ and 0.583 for $\omega = 0.76$ as compared with values of 0.587 and 0.597 in Ref. 1. It appears to the author that perhaps Chapman's value of 0.587 may not be accurate to the third place as indicated, since his velocities in the outer region deviate from those given by Eq. (6) even though both solutions begin at essentially the same ζ_n.

Figure 3 shows the variation in ζ_n with M_e for several values of ω as evaluated from the results of Table 1. The variation in ζ_δ would be similar since λ does not change substantially (Table 1). These results indicate the possibility of a substantial effect of ω on the pressures in base regions where a net gas outflow or inflow exists. This effect can be readily computed. For no net outflow or inflow, however, ω would have little effect, since u_D^* is essentially constant.

Lykoudis[2] considered the possibility of introducing a Reynolds number effect on u_D^* at the neck of the wake in base regions by assessing the necessary adjustment in the boundary condition at the outer edge of the viscous layer. He assumed that $\zeta_n \to \infty$, but essentially showed by an approximate analysis that u^* is near zero at $\zeta_n > 3.0$. It appears that his conclusions are essentially correct even though ζ_n is always considerably less than 3.0. His empirical relation for the neck height y_n can be rearranged to give

$$y_n(u_e/x\nu)^{1/2} = 40/(\tau \cos\beta)^{1/2} \quad (11)$$

where β is the wake angle and τ is the ratio of the distance of the neck back from the base divided by the base radius. Considering his case of $M_e = 0$ (or $\omega = 1.0$), Eq. (6) gives the velocity distribution shown on Fig. 4. Clearly, u^* is essentially zero for $y(u_e/x\nu)^{1/2} > 15$. Since $\cos\beta$ is near 1.0, it follows from Eq. (11) that τ must be of the order of 10 before the outer boundary condition is substantially influenced.

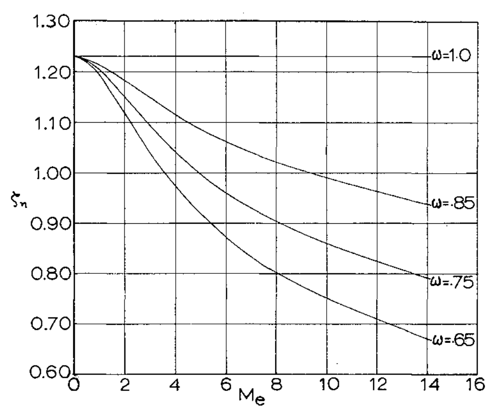

Fig. 3 Effect of freestream Mach number and viscosity-temperature relation on ζ_n.

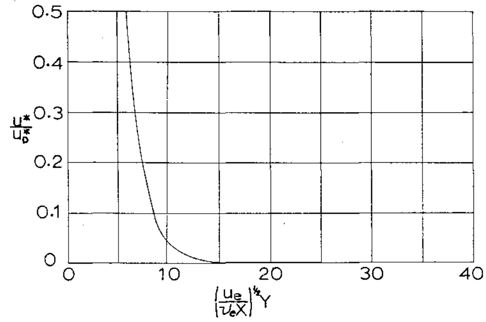

Fig. 4 Velocity profile in the physical coordinate ($\omega = 1.0$).

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Creep and Relaxation

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In performing analyses in viscoelasticity, it is often necessary to convert a known relaxation modulus $E(t)$ to its associated creep compliance $J(t)$ and vice versa. Hopkins and Hamming[1] have presented a step-by-step numerical integration method for obtaining one material function when the other is known. In their computations, the interrelationship between $E(t)$ and $J(t)$ is taken in the form

$$\int_{-\infty}^{t} E(t - t')\, J(t')\, dt' = t\, H(t) \tag{1}$$

where $H(t)$ is the Heaviside unit step function. The role of $E(t)$ and $J(t)$ may be interchanged in (1). Utilizing this form of the interconversion, it is necessary to define the integral of one of the functions in order to perform the numerical integration. If the interrelationship between $E(t)$ and $J(t)$ is taken in the alternative form

$$\int_{-\infty}^{t} E(t - t')\, \frac{dJ(t')}{dt'}\, dt' = H(t) \tag{2}$$

numerical integration may be performed without introducing the integral of one of the functions. To this end, the discontinuity term in (2) is removed, and (2) is integrated by parts, yielding

$$J(t)\, E(0) - \int_{0}^{t} J(t')\, \frac{dE(t - t')}{dt'}\, dt' = 1 \tag{3}$$

Using the step-by-step integration procedure proposed by Lee and Rogers,[2] Eq. (3) may be solved for $J(t_n)$, where t_n corresponds to the present time t. As before, if it is desired to determine $E(t)$ from a known $J(t)$, the role of E and J may be interchanged in (3). As stated in Ref. 2, an advantage is gained by using this form of the material property interrelationship since the form of the time step becomes arbitrary and may be logarithmic, constant, etc.

If secondary creep exists in $J(t)$, the convergence requirements for numerical integration may break down, and the solution may become unstable. To circumvent this difficulty, the secondary creep is removed by defining

$$J(t) = S(t) + Rt \tag{4}$$

where $J(0) = S(0)$ and $R = \dot{J}(\infty)$. A $(\dot{\ })$ denotes differentiation with respect to time. Thus, in (4), $S(t)$ is now a bounded function for all time. Substitution of (4) into (3) yields, after simplification,

$$S(t)\, E(0) - \int_{0}^{t} S(t')\, \frac{dE(t - t')}{dt'}\, dt' = 1 - R \int_{0}^{t} E(t')\, dt' \tag{5}$$

The constant R, i.e., $\dot{J}(\infty)$, may be evaluated by integrating the left-hand side of (5) by parts and taking the limit as t goes to infinity. Upon noting that, for secondary creep to be present, $E(\infty)$ is zero, and by the definition of $S(t)$, $S(\infty)$ is zero, the integral term may be shown to vanish, and

$$R = 1 \Big/ \int_{0}^{\infty} E(t)\, dt \tag{6}$$

Care must be exercised in using (6) as it is valid only if the secondary creep is present; otherwise, R is identically zero.

Step-by-step numerical integration of (5) by the scheme proposed in Ref. 2 yields

$$\tfrac{1}{2}\{E(0) + E(t_n - t_{n-1})\}\, S(t_n) = 1 - R\, F(t_n) + \tfrac{1}{2}\, J(0)\, \{E(t_n - t_1) - E(t_n - t_{i-1})\} + \frac{1}{2}\sum_{i=1}^{n-1} J(t_i)\, \{E(t_n - t_{i+1}) - E(t_n - t_{i-1})\} \qquad n > 1 \tag{7}$$

where

$$F(t_n) = \int_{0}^{t_n} E(t')\, dt' \tag{8}$$

For $t = 0$ and $t = t_1$,

$$S(0) = J(0) = 1/E(0)$$

and

$$S(t_1) = \frac{S(0)\, [3\, E(0) - E(t_1)] - 2\, R\, F(t_1)}{E(0) + E(t_1)} \tag{9}$$

Finally, the creep compliance is computed from (4) as follows:

$$J(t_n) = S(t_n) + Rt_n \tag{10}$$

If no secondary creep exists, all terms multiplied by R vanish, and $S(t_n)$ is identically equal to $J(t_n)$.

As an example, the creep compliance is calculated from (7) for a material whose relaxation function is given by

$$E(t) = E_0 \left\{ \frac{1}{6} \int_{t}^{te^6} \frac{e^{-u}}{u}\, du \right\} \tag{11}$$

This relaxation function results from a problem defined in Refs. 3 and 4, and in Sec. 12 of Ref. 5 The numerical inversion is accomplished by direct inversion without removing the secondary creep and by the modified procedure presented herein. Constant logarithmic steps of 0.05 were used for both inversions. The results are shown in Fig. 1. The results are compared to the exact solution obtained from Ref. 4. As is seen from Fig. 1, direct numerical inversion yields results that are considerably in error from the exact solution, whereas the modified solution obtained by first removing the secondary creep agrees very well when compared to the exact solution (for the scale used in Fig. 1, no difference is visible).

This note has presented a numerical method for determining the creep compliance (when secondary creep is present) from its associated relaxation modulus. The numerical example presented indicates the error that can result from a direct inversion without first removing the secondary creep.

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